B.Sc (Maths) - part II

paper III

Topic:- Concept of subgroup and cyclic group with Example

Subgroup :- Def Let G be a group and H a subset of G. Then it is called to be subgroup of G if H is group under group operation of G.

Example :- The set E of even integer is a subgroup of group Z of integer under ordinary addition.

Soln:- Clearly the set Z of integers is a group under ordinary addition, 0 being the identity element and $-a \in Z$ being the inverse element of $a \in Z$.

Now $b \in E$ means $b = 2a$ where $a \in Z$ clearly $E \subset Z$

Also, it can easily be verified that E is a group under ordinary addition, 0 being the identity element and $-b \in E$ being the inverse element of $b \in E$

Thus $E \subset Z$ and E is a group under

the same operation as that on Z.

Cyclic group :- Def:- A group (G, o) is called cyclic group if there exists an element $a \in G$ such that every element b of G can be expressed as an integral (positive & negative) power of a. The element a whose power exhaust the set G is called a generator of the cyclic group.

Thus a cyclic group consists only of integral power of a particular element of the group.

Example:- The set of fourth roots of unity form a cyclic group under multiplication of complex number.

The set of fourth roots of unity is $G = \{1, -1, i, -i\}$ we know G is a group under the multiplications of complex numbers. 1 being the identity

element and inverse element being $1, -1, -i, i$ respectively

Now $(i)^1 = i$, $(i)^2 = -1$ etc

$(-1)^1 = -1$, $(-1)^2 = 1$, $(-1)^3 = -1$ etc

$(i)^1 = i$, $(i)^2 = -1$, $(i^3) = -i$, $(i)^4 = 1$

$(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$,

$(-i)^4 = 1$

Thus we find the integral power of i exhaust are the elements of G. $SO(G, \cdot)$ is a cyclic group with i as generator.

Theorem:- To prove that a non empty subset H of the group G under the operation $\circ$ forms a subgroup if and only if

$$a, b \in H \Leftrightarrow a \circ b^{-1} \in H$$

proof:- Let H be a subgroup of the group G under the operation '$\circ$' then $H \subseteq G$ and H is group under $\circ$.

$\therefore b \in H \Rightarrow b^{-1} \in H$ ($\because$ inverse element exists in a group)

$\therefore a, b \in H \Rightarrow a \in H, b^{-1} \in H$

$\Rightarrow a \circ b^{-1} \in H$ (closure law hold in group)

$\therefore$ The condition is necessary

Next let $H \subseteq G$ and $a, b \in H \Rightarrow a * b^{-1} \in H$ ——— (1)
we have to prove that H is a subgroup of G i.e H is group under the operation $*$.

Now $a \in H, a^{-1} \in H \Rightarrow a * a^{-1} \in H$ by (1)
$\Rightarrow e \in H$
$\therefore$ The identity element exists in H.

Again $e \in H, a \in H \Rightarrow e * a^{-1} \in H$ by (1)
$\Rightarrow a^{-1} \in H$
$\therefore$ Inverse elements of the elements of H exists in H ——— (2)

Now $a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$ by (2)
$\Rightarrow a * (b^{-1})^{-1} \in H$ by (1)
$\Rightarrow a * b \in H$
$\therefore$ The closure law holds in H

Also the associative law holds in H because $H \subseteq G$ and the associative law holds in G because it is group
Hence H satisfies itself under operation $*$.

Thus H is subgroup of G under the operation $*$.

$\therefore$ The condition is also sufficient.